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EQUATIONS OF A SIMPLE FLAME SOLVED BY SUCCESSIVE APPROXIMATIONS TO THE SOLUTION OF AN INTEGRAL EQUATION *

(PART II: SECOND ORDER REACTION)

G. Klein

ABSTRACT

The problem of an idealized flame whose underlying chemical reaction is unimolecular and reversible (where the kinetic energy of the gas stream is neglected), which has been solved by an integral equation method of successive approximations for a first reaction in PART I, is now extended to a second order reaction. This problem is very nearly equivalent to that of a simple chain reaction flame in which the catalyst reaction(s) are assumed to be in equilibrium. In this case the behavior near the hot boundary of the functions involved is very different from that of the case of a first order reaction, and a careful choice of the integral equation and of the lowest approximation to be adopted has to be made. - The diffusion coefficient is assumed constant: for a certain value of this constant the problem simplifies considerably and for other values of an alternative perturbation and expansion method is proposed which involves only linear differential equations. - It is verified that neglect of the back reaction affects the method and results immaterially, and the effect of varying ~~of~~ the hot boundary temperature is briefly considered.

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Series 8
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by G. Klein

which will be referred to as

(PART I: FIRST ORDER REACTION)

REFERENCES

1) J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, Molecular Theory of Gases and Liquids. Wiley, (1954). Chapter 11.7.

2) C. F. Curtiss, J. O. Hirschfelder, and D. E. Campbell, The Theory of Flame Propagation and Detonation, III, University of Wisconsin Naval Research Laboratory Report, 15 Feb. 1952 (where a detailed numerical solution of the problem is given in the appendix); Reprinted without appendix in 4th Int. Symp. for Combustion, p. 190, Publ. by Williams and Wilkins (1953).

1. INTRODUCTION

The problem of an idealized flame whose underlying chemical reaction is unimolecular, reversible, and of the first order, which has already been treated and solved in the references quoted below, is reconsidered here (kinetic energy of the gas stream being neglected). Its solution is made to depend on the solution of an integral equation which contains an unknown parameter whose eigenvalue has to be determined. This equation is solve by a method of successive approximations.

Except for minor and obvious deviations,* the notation is the same as that used in the first reference quoted; equations there are referred to on the left margin.

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* The only ones being (cf. equations 1.1-1.3)

(11.7-27) $\frac{m}{g}\frac{d}{d\tau} = \frac{d}{d\zeta}$ (-eliminating the distance variable)

(11.7-25) $R(x,\tau) = -f(x,\tau)$ (-an essentially positive quantity)

(11.7-31) $\delta = \frac{\gamma}{1-\gamma}\frac{1}{\beta}$ (-for conciseness)

Flame equations. These are, in terms of dimensionless variables and parameters, the equations of continuity (or chemical reaction), diffusion, and energy (or thermal conduction),

(11.7-25) $$g \frac{dG}{d\tau} = -\frac{1}{\mu^2} R(x, \tau) \quad ,$$ 1.1

(11.7-28) $$g \frac{dx}{d\tau} = \frac{1}{\delta} (x - G) \quad ,$$ 1.2

(11.7-31 $$g = \frac{1}{b} (G - G_\infty) - (\tau_\infty - \tau)$$ 1.3

Hot Boundary conditions. At the hot boundary chemical reaction, diffusion and thermal conduction cease. Thus

(11.7-32) $$R(x_\infty, \tau_\infty) = 0 \quad ,$$ 1.4

(11.7-34) $$x_\infty = G_\infty$$ 1.5

Equation 1.3 is the integrated energy balance equation, and by suitable choice of the constant of integration the third boundary condition, that the temperature gradient must vanish,

$$g(\tau_\infty) = 0 \quad ,$$ 1.6

has already been taken care of.

Cold boundary conditions. If one assumes a conventional functional form for the reaction rate, where the latter does not actually vanish at the cold boundary temperature, some care is needed in the stipulation of the cold boundary conditions. Experimentally, however, and in computation where in any case one confines oneself to a limited number of decimal places, the reaction rate can be taken as zero at and near the cold boundary temperature. Thus in practice there is no doubt what the conditions should be, they are analogous to those at the hot boundary, viz.

$$R(x_0, \tau_0) = 0 \quad ,$$ 1.7

(11.7-35) $$x_0 = G_0 = 1 \quad ,$$ 1.8

$$g(\tau_0) = 0$$ 1.9

Auxiliary quantities. It is convenient to define the known linear function

$$x^* = x_\infty + b(\tau_\infty - \tau) \quad ,$$ 1.10

and the parameter 1.11

$$q = \frac{1}{b\mu^2}$$

Elimination of the mass rate of flow. We consider the temperature gradient and the concentration as the primary dependent variables. From 1.3, 1.5, and 1.10,

$$G = x^* + bq \quad , \qquad 1.12$$

so that if the temperature gradient is known, the fractional mass rate of flow, G, can be readily found.

Fundamental simultaneous equations. With 1.12, 1.10 equations 1.1, 1.2 may be written in the form

$$\boxed{\begin{aligned} q\left(1 - \frac{dq}{d\tau}\right) &= q_0 R(x, \tau) \quad , & 1.13 \\ x &= x^* + \left(b + \delta \frac{dx}{d\tau}\right) q & 1.14 \end{aligned}}$$

These equations have to be satisfied simultaneously, the solutions being subject to the boundary conditions. It should be noted that this is an eigenvalue problem; the parameter q_0 in 1.13 is not known and depends on the boundary conditions.

Special cases. In the following two cases the problem simplifies considerably:

When $\delta = 1$, it is clear from 1.10 that 1.14 is satisfied by

$$x_{\delta=1} = x^* \quad , \qquad 1.15$$

and hence the problem reduces to the solution of the single differential equation

$$q_{\delta=1}\left(1 - \frac{dq_{\delta=1}}{d\tau}\right) = q_{0\,\delta=1} \, R(x^*, \tau) \quad . \qquad 1.16$$

When $\delta = 0$, equation 1.14 gives

(11.7-44) $$x_{\delta=0} = x^* + bq_{\delta=0} \quad , \qquad 1.17$$

which when substituted into 1.13 again leads to a single differential equation,

$$q_{\delta=0}\left(1 - \frac{dq_{\delta=0}}{d\tau}\right) = q_{0\,\delta=0} \, R(x^* + bq_{\delta=0}, \tau) \qquad 1.18$$

This latter equation simplifies further if the reaction rate is linear in the fuel gas concentration.

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(PART I: FIRST ORDER REACTION)

List of Errata

On cover and title page add: (PART I: FIRST ORDER REACTION)

P. 1, first equation in the footnote should read: $g \frac{d}{d\tau} = \frac{d}{d\zeta}$

P. 7, last symbol in equation 2.4 should be τ_o

P. 7, first factor on the right of equation 2.5 should be $(\tau_a - \tau_o)$

P. 7, in the line before eqn. 2.8 the reference should be to equ. 1.21

P. 9, graph 4: the point $q_{\xi=1}^{(\nu=5)} = 398.5$ should have been marked

LIST OF SYMBOLS

(in the order in which they occur)

p. 25	τ $\mathfrak{z}$ ζ q R x x^*
	δ $\mathfrak{b}$ τ_∞ x_∞ τ_0 x_0
p. 26	τ_∞^* ξ $\mathcal{R}_i$ $\mathfrak{R}$
p. 28	ψ
p. 30	m θ t r
p. 31	ν N
p. 32	I
p. 34	σ
p. 37	ε
p. 40	A B C φ $\ddot{\Phi}$
p. 41	κ

7. REVISION: Fundamental simultaneous differential equations; Boundary conditions; Auxiliary functions; Summary

In the simple flame problem considered in PART I we have adopted the reduced temperature τ as independent variable and the temperature gradient g as the main dependent variable: these two are related to the reduced distance ζ by

$$g = \frac{d\tau}{d\zeta} . \qquad 7.1$$

It has been shown (cf. 1.13, 1.14, 1.10) that the problem of a simple unimolecular flame is contained in the equations

$$(1.13) \qquad g\left(1 - \frac{dg}{d\tau}\right) = q\,R(x,\tau) , \qquad 7.2$$

$$(1.14) \qquad x - x^* = \left\{(1-\delta)b + \delta\,\frac{d(x - x^*)}{d\tau}\right\} g \qquad 7.3$$

together with the boundary conditions

$$(1.6) \qquad x(\tau_\infty) = x_\infty , \qquad g(\tau_\infty) = 0 , \qquad 7.4$$

$$(1.4) \qquad R(x_\infty, \tau_\infty) = 0 , \qquad 7.5$$

$$(1.9) \qquad x(\tau_0) = x_0 , \qquad g(\tau_0) = 0 , \qquad 7.6$$

where it is assumed that the total reaction rate $R(x,\tau)$ for practical purposes vanishes at and near the cold boundary τ_0.

The function x^* in 7.3 is linear in τ and coincides with the fuel component x at the boundaries, thus

$$\frac{x^* - x_\infty}{x_0 - x_\infty} = \frac{\tau_\infty - \tau}{\tau_\infty - \tau_0} . \qquad 7.7$$

Also,

$$b = \frac{x_0 - x_\infty}{\tau_\infty - \tau_0} , \qquad 7.8$$

and previously we had written (cf. 1.10)

$$(1.10) \qquad x^* = x_\infty + b(\tau_\infty - \tau) . \qquad 7.9$$

It will be useful to define a fictitious beyond-the-hot-boundary temperature $\tau_\infty^* > \tau_\infty$, where

$$\chi_\infty = b(\tau_\infty^* - \tau_\infty) , \qquad 7.10$$

so that 7.7 may be written more concisely as

$$\chi^* = b(\tau_\infty^* - \tau) , \qquad 7.11$$

and we note

$$\chi^*(\tau_0) = \chi_0 \quad ; \qquad \chi^*(\tau_\infty) = \chi_\infty \quad , \qquad 7.12$$

$$\chi^*(\tau_\infty^*) = 0 \ . \qquad 7.13$$

It has already been remarked (cf. 1.15) , and it is easily verified by inspection of the equation, that in the case $\delta = 1$ a solution of 7.3 is $\chi = \chi^*$; in this case the right hand member of 7.2 does not contain g : the function $R(\chi^*, \tau)$ is wholly known. This, and the form of equation 7.3 suggest adopting the new variable

$$\xi = \chi - \chi^* , \qquad 7.14$$

replacing χ , and expanding

$$R(\chi, \tau) = \mathcal{R}_0(\tau) + \mathcal{R}_1(\tau)(\chi - \chi^*) + \mathcal{R}_2(\tau)(\chi - \chi^*)^2 + \cdots \qquad 7.15$$

Here

$$\mathcal{R}_0(\tau) = R(\chi^*, \tau) , \qquad 7.16$$

and the other functions $\mathcal{R}_1(\tau)$, $\mathcal{R}_2(\tau)$, . . . , are similiarly known if the form of the reaction rate is given. We write

$$R(\chi, \tau) = \mathcal{R}(\xi, \tau) . \qquad 7.17$$

Cf. p. 11!

Summary. In terms of the functions just defined we have

$$\mathcal{R}(\xi,\tau) = \mathcal{R}_0(\tau) + \mathcal{R}_1(\tau)\xi + \mathcal{R}_2(\tau)\xi^2 + \dots , \qquad 7.18$$

and the fundamental equations 7.1-7.3 may be written

$$\boxed{\begin{aligned} g\left(1-\frac{dg}{d\tau}\right) &= g\,\mathcal{R}(\xi,\tau) && 7.19\\ \frac{d\zeta}{d\tau} &= \frac{1}{g} && 7.20\\ \delta\frac{d\xi}{d\tau} - \frac{1}{g}\xi &= -(1-\delta)b && 7.21 \end{aligned}}$$

The boundary conditions 7.4-7.6 have now, in view of 7.14, 7.12, the simple form

$$\xi(\tau_\infty) = 0 \quad ; \qquad g(\tau_\infty) = 0 , \qquad 7.22$$

$$\mathcal{R}_0(\tau_\infty) = 0 , \qquad 7.23$$

$$\xi(\tau_0) = 0 \quad , \qquad g(\tau_0) = 0 , \qquad 7.24$$

and it is understood that for practical purposes the functions $\mathcal{R}_0(\tau)$, $\mathcal{R}_1(\tau)$, $\mathcal{R}_2(\tau)$, . . . , vanish at and near the cold boundary τ_0 .

Alternative summary in which 7.20, 7.21 are written in integrated form:

$$\boxed{\begin{aligned} g\left(1-\frac{dg}{d\tau}\right) &= g\,\mathcal{R}(\xi,\tau) ; \quad g(\tau_0)=0,\ g(\tau_\infty)=0 && 7.25\\ \zeta &= \int^{\tau}\frac{1}{g}\,d\tau && 7.26\\ \xi &= \frac{1-\delta}{\delta}\,b\, e^{\frac{1}{\delta}\zeta}\int_{\tau}^{\tau_\infty} e^{-\frac{1}{\delta}\zeta}\,d\tau && 7.27 \end{aligned}}$$

where

$$\mathcal{R}(\xi,\tau) = \mathcal{R}_0(\tau) + \mathcal{R}_1(\tau)\xi + \mathcal{R}_2(\tau)\xi^2 + \dots , \qquad \mathcal{R}_0(\tau_\infty) = 0 \qquad 7.28$$

and it is understood that the $\mathcal{R}_i(\tau)$, which are known functions, for practical purposes vanish at and near the cold boundary τ_0 .

8. REACTION RATE AND NATURE OF THE SOLUTION

First order reaction. When the reaction is of the first order,

$$\mathcal{R}(\xi, \tau) = \mathcal{R}_0(\tau) + \mathcal{R}_1(\tau)\,\xi .$$

This case has been dealt with in Part I where

$$\mathcal{R}_1(\tau) = \Psi(\tau) , \qquad (1.21)$$

$$\mathcal{R}_0(\tau) = R(x^*, \tau) .$$

The total reaction rate $\mathcal{R}(\xi,\tau)$ has the single zero at τ_∞ where

$$\mathcal{R}_0(\tau_\infty) = 0 .$$

Second order reaction. Here

$$\mathcal{R}(\xi, \tau) = \mathcal{R}_0(\tau) + \mathcal{R}_1(\tau)\,\xi + \mathcal{R}_2(\tau)\,\xi^2 . \qquad 8.1$$

We shall consider the special case where the presence of the fuel is essential for both the forward and the back reaction: the fuel acts at the same time as a catalyst.* Thus (cf. 7.14-7.17)

$$R(0, \tau) = 0 , \qquad \text{i.e.} \qquad \mathcal{R}(-x^*, \tau) = 0 . \qquad 8.2$$

Hence, by 7.15, or 8.1,

$$\mathcal{R}_0(\tau) = x^*\left\{\mathcal{R}_1(\tau) - x^*\,\mathcal{R}_2(\tau)\right\} \qquad 8.3$$

This implies that $\mathcal{R}_0(\tau)$ has two zeros, one at τ_∞, by 7.23,

$$\mathcal{R}_0(\tau_\infty) = 0 , \qquad 8.4$$

and the other, due to the first factor on the right of 8.3 (cf. 7.11) at τ^*_∞,

$$\mathcal{R}_0(\tau^*_\infty) = 0 . \qquad 8.5$$

By 7.12 and 8.4,

$$x_\infty = \frac{\mathcal{R}_1(\tau_\infty)}{\mathcal{R}_2(\tau_\infty)} \qquad 8.6$$

The full reaction rate $\mathcal{R}(\xi,\tau)$ will also, in general, have two zeros, one at τ_∞, and another beyond but near to τ_∞; it will be at τ^*_∞ in the case $\delta = 1$.

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* cf. APPENDIX, p. 40

Nature of the solution. The nature of the required solution for any given δ (where δ is of order unity) does not differ drastically from that of the case $\delta = 1$, so that we confine ourselves to a short discussion of the latter. We have

$$g\left(1 - \frac{dg}{d\tau}\right) = q\,\mathcal{R}_0(\tau) \,, \qquad 8.7$$

$$\mathcal{R}_0(\tau_0) = 0 \,, \left[\frac{d}{d\tau}\,\mathcal{R}_0(\tau)\right]_{\tau_0} = 0 \,; \quad \mathcal{R}_0(\tau) > 0 \,, \quad \tau_0 < \tau < \tau_\infty \,; \qquad 8.8$$

$$\mathcal{R}_0(\tau_\infty) = 0 \,, \qquad \mathcal{R}_0(\tau_\infty^*) = 0 \,, \qquad 8.9$$

and the required solution must satisfy

$$g(\tau_0) = 0 \,, \qquad g(\tau_\infty) = 0 \qquad 8.10$$

The tangent elements of the family of integral curves of 8.7 are easily plotted from the relation

$$\frac{dg}{d\tau} = 1 - \frac{q\,\mathcal{R}_0}{g} \qquad 8.11$$

The locus of stationary points of the integral curves is evidently the curve $q\,\mathcal{R}_0(\tau)$; the τ -axis is the locus of points at which there is infinite slope; and the lines $\tau = \tau_\infty$, $\tau = \tau_\infty^*$, are loci of points at which there is gradient unity. There are <u>two</u> singular points, viz. on the τ -axis at $\tau = \tau_\infty$ and at $\tau = \tau_\infty^*$. The former is a saddle point and the required solution is the singular integral which passes through it with negative slope.

The parameter q in 8.7 is not known: instead there are two boundary conditions 8.10, i.e., q is an eigenvalue parameter. The nature of the second singular point at τ_∞^* depends on the magnitude of q - in the numerical application considered it is a spiral point (See graph 12.)

Series expansion near τ_∞. Since $g(\tau_\infty)=0$, $R_0(\tau_\infty)=0$, one finds easily from 8.7

$$\left[-\frac{dg}{d\tau}\right]_{\tau_\infty} = \frac{1}{2}\left\{\sqrt{4q\left[-\frac{dR_0}{d\tau}\right]_{\tau_\infty}+1}\;-1\right\}, \qquad 8.12$$

$$\left[\frac{d^2g}{d\tau^2}\right]_{\tau_\infty} = q\left[\frac{d^2R_0}{d\tau_2}\right]_{\tau_\infty}\Bigg/\left(1+3\left[-\frac{dg}{d\tau}\right]_{\tau_\infty}\right), \qquad 8.13$$

etc. from which a series expansion near τ_∞ may be obtained.

Solution near τ_∞^* in polar coordinates. Near τ_∞^* we put

$$qR_0(\tau) = q\left[\frac{dR_0}{d\tau}\right]_{\tau_\infty^*}(\tau-\tau_\infty^*) = m(\tau-\tau_\infty^*), \qquad 8.14$$

so that equation 8.7 becomes

$$\frac{dg}{d\tau} = \frac{g-m(\tau-\tau_\infty^*)}{g} \qquad 8.15$$

On changing to polar coordinates

$$g = r\sin\theta, \qquad 8.16$$

$$\tau-\tau_\infty^* = r\cos\theta, \qquad 8.17$$

and writing

$$t = \tan\theta, \qquad 8.18$$

one obtains the solution near τ_∞^* in the form

$$r = r_{-\pi/2}\left(\frac{t^2+1}{t^2-t+m}\right)^{\frac{1}{2}} e^{\frac{1}{2}(m-\frac{1}{4})^{-\frac{1}{2}}\left(-\frac{\pi}{2}-\tan^{-1}\frac{t-\frac{1}{2}}{\sqrt{m-1/4}}\right)}. \qquad 8.19$$

That is, it is a spiral provided

$$4q\left[\frac{dR_0}{d\tau}\right]_{\tau_\infty^*} > 0 \qquad 8.20$$

The nature of $g(\tau)$ beyond τ_∞ is of course of no physical interest.

9. INTEGRAL EQUATIONS AND METHOD OF SUCCESSIVE APPROXIMATIONS

In order to solve equations 7.25-7.27 we employ a method of successive approximations as follows. Suppose a ν -th approximation, $g^{(\nu)}$, to g is known: then by 7.26 and 7.27 we find the corresponding approximations to ζ and ξ thus:

$$\zeta^{(\nu)} = \int^{\tau} \frac{1}{g^{(\nu)}} \, d\tau \quad , \tag{9.1}$$

$$\xi^{(\nu)} = \frac{1-\delta}{\delta} \, k \, e^{\frac{1}{\delta}\zeta^{(\nu)}} \int_{\tau}^{\tau_\infty} e^{-\frac{1}{\delta}\zeta^{(\nu)}} \, d\tau \, . \tag{9.2}$$

Equation 7.25 is the hard core of the problem. It is turned into an integral equation - but this can be done in an infinite number of ways: for instance, multiplication of 7.25 by g^N, where N is not necessarily an integer, and integration, result in

$$\frac{1}{N+2} g^{N+2} = q \int_{\tau}^{\tau_\infty} g^N \mathcal{R}(\xi, \tau) \, d\tau - \int_{\tau}^{\tau_\infty} g^{N+1} d\tau \, . \tag{9.3}$$

One could now multiply 9.3 by some suitable function of N and add similar equations for all integers N starting from $N = 0$, and would thus obtain an integral equation in well-known analytic functions of g . However, we have not succeeded in determining which functions would be the most 'natural' ones to take, and therefore confine ourselves to an integral equation of the form 9.3 for a suitable N.

In 9.3 the upper boundary condition has been taken care of; the lower boundary condition fixes the value of the eigenvalue parameter, viz.,

$$q = \int_{\tau_0}^{\tau_\infty} g^{N+1} d\tau \Bigg/ \int_{\tau_0}^{\tau_\infty} g^N \mathcal{R}(\xi, \tau) \, d\tau \, . \tag{9.4}$$

Hence the integral equation 9.3 may be written in the form

$$g = \left[(N+2) \int_{\tau}^{\tau_0} g^{N+1} d\tau \left\{ \frac{\int_{\tau}^{\tau_0} g^N R(\xi,\tau)\, d\tau}{\int_{\tau_0}^{\tau_0} g^N R(\xi,\tau)\, d\tau} - \frac{\int_{\tau}^{\tau_0} g^{N+1} d\tau}{\int_{\tau_0}^{\tau_0} g^{N+1} d\tau} \right\} \right]^{\frac{1}{N+2}} \qquad 9.5$$

which does not contain the eigenvalue parameter and includes both boundary conditions.

Let us write 9.5 more concisely as

$$g = I(g, \xi, \tau; N) \qquad 9.6$$

In fact, of course, the right hand side of 9.5 or 9.6 is not a function of N at all. Again, on the right of 9.5 or 9.6 the quantities g and ξ occur only inside integrals which one may assume not to be too sensitive if instead of g and ξ approximate values of these are substituted. Thus the method of successive approximations is contained in the equations 9.1, 9.2, and

$$g^{(\nu+1)} = I(g^{(\nu)}, \xi^{(\nu)}, \tau; N) \qquad 9.7$$

The method is clearly justified if the $g^{(\nu)}$ and $\xi^{(\nu)}$ form convergent sequences.

The most 'natural' values for N which suggest themselves are $N = -1$ and $N = 0$ which amount to integration of 7.25 at sight. The choice of the lowest approximation $g^{(0)}$ will be discussed in the following section.

10. DISCUSSION; CHOICE OF LOWEST APPROXIMATION

The solution of the flame equations by the method of successive approximations depends on the choice of a suitable recurrence relation based on 9.3, like 9.7, and on the choice of a suitable lowest approximation $g^{(0)}$.

The exponent N . As to the first problem, once relation 9.7 is adopted, the following comments on the choice of the exponent are relevant:

$N = -1$. This case follows by dividing 7.25 by g and integrating; it has been employed in the first order reaction problem of Part I. Advantage: The second ratio on the right of 9.5 is the same for any particular τ at any stage of the operation 9.7. Disadvantages: a) expressions of the form $0/0$ occur in the first ratio and these have to be evaluated separately. b) for an unsuitable lowest approximation a supposed $g^{(\nu+1)}$ may become negative, which entrains an infinite integrand in the succeeding stage - the method breaks down.

$N = 0$. This case follows by integrating 7.25 at sight. Advantages: a) no ratios of the form $0/0$ occur, b) for $\delta = 1$ the first ratio on the right of 9.5 is the same for any particular τ at all stages of the operation 9.7. Disadvantage : for an unsuitable first approximation a supposed $g^{(\nu+1)}$ may become imaginary:- the method breaks down.

$N = 1$. For an unsuitable first approximation a supposed $g^{(\nu+1)}$ may become negative, and succeeding $g^{(\nu)}$ at the same τ remain negative - the method breaks down.

$N = \frac{1}{2}$. At first sight none of the more obvious possibilities of a direct break-down of the method, of the kind just described, seem to arise in this case.

Lowest approximation. It remains to choose a lowest approximation $g^{(0)}$ for the operations 9.7. We consider in detail the case $\delta = 1$ so that g is a solution of

$$g\left(1 - \frac{dg}{d\tau}\right) = q\,\mathcal{R}_0(\tau) \,. \qquad 10.1$$

It is desirable that the lowest approximation should already possess as many properties as possible of the actual solution; the properties of g which follow from 10.1 without further quantitative knowledge of $\mathcal{R}_0(\tau)$ are (cf. 8.7-8.11), with increasing τ,

1) Zero at $\tau = \tau_0$
2) Gradient unity at $\tau = \tau_0$
3) Maximum occurs on $q\,\mathcal{R}_0(\tau)$
4) Zero at $\tau = \tau_\infty$
5) Minimum occurs on $q\,\mathcal{R}_0(\tau)$
6) Gradient unity at $\tau = \tau_\infty^*$
7) Further zero at $\tau = \tau_\infty^* + \sigma$ (say, where σ is small and positive)
8) Gradient infinite at $\tau = \tau_\infty^* + \sigma$ (curve convex to increasing τ)
9) Spiral around $\tau = \tau_\infty^*$.

Since q is not known, 3) and 5) are unsuitable for inclusion in any first approximation; similarly, 9) is unsuitable. The remaining conditions cannot be catered for by a polynomial but they are embodied in the following algebraic curve

$$g^{(0)}(\tau) = \frac{1}{(\tau_\infty - \tau_0)\sqrt{\tau_\infty^* + \sigma - \tau_0}}\,(\tau - \tau_0)(\tau_\infty - \tau)\sqrt{(\tau_\infty^* + \sigma) - \tau} \,, \qquad 10.2$$

where σ is determined so that

$$\left[\frac{d g^{(0)}(\tau)}{d\tau}\right]_{\tau = \tau_\infty^*} = 1 \,, \qquad 10.3$$

and one finds that σ must be a solution of

$$16\left(\frac{\sigma}{\tau_\infty^* - \tau_\infty}\right)^2 - 4\left\{\left(\frac{\tau_\infty^* - \tau_0}{\tau_\infty^* - \tau_\infty}\right)^2 - \left(\frac{\tau_\infty^* - \tau_0}{\tau_\infty^* - \tau_\infty}\right) + 2\right\}\left(\frac{\sigma}{\tau_\infty^* - \tau_\infty}\right) + \left(\frac{\tau_\infty^* - \tau_0}{\tau_\infty^* - \tau_\infty}\right) = 0 \qquad 10.4$$

See graph 13, p.45

<u>Discussion and results.</u> 1. For $\delta = 1$ formula 9.7 reduces to

$$g^{(\nu+1)} = I(g^{(\nu)}, \tau; N) \qquad 10.5$$

We took $g^{(0)}$ as in 10.2 and $N=0$, and performed the operation 10.5 up to $\nu = 10$. Result: The bulk of the $g^{(\nu)}$ converge; near τ_∞ they diverge, and at the stage $\nu = 10$ the method breaks down. - the integrations were performed by the trapezoidal rule and the last interval adjoining τ_∞ was about $(\tau_\infty - \tau_0)/400$; the divergence was oscillatory about fairly definite 'mean' values. When these values were substituted on the right of 10.5 they very nearly reproduced themselves on the left. This suggested use of the formula

$$g^{(\nu+1)} = I\left(\frac{g^{(\nu)} + g^{(\nu-1)}}{2}, \tau; N\right), \qquad 10.6$$

instead of 10.5, from which we finally obtained 'output' g's agreeing with the 'input' g's to any required number of significant digits.

2. For $\delta = 3/4$ we took $g^{(0)} = g_{\delta=1}$, $N = 0$, and used a corresponding procedure.

3. Similarly, for $\delta = 1/2$, where we started with $g^{(0)} = g_{\delta=3/4}$.

4. The attempt of solving the case $\delta = 0$ in a similar manner leads to an immediate break-down of the method when $N = 0$. We obtained a solution as follows: Extrapolation from $g_{\delta=1}, g_{\delta=3/4}, g_{\delta=1/2}$ gives an approximation to g; this value is used to calculate a parabolic approximation to g near τ_∞; the latter is combined with an approximation to g, extrapolated from $g_{\delta=1}, g_{\delta=3/4}, g_{\delta=\frac{1}{2}}$, for the greater part of the range $\tau_0 \rightarrow \tau_\infty$; this 'mixed' lowest approximation is now put through 9.1, 9.2, 9.7 (modified as in 10.6).

We have thus found solutions for the cases $\delta = 1, 3/4, 1/2, 0$ which satisfy the flame equations: this is all that is required for practical purposes but the procedure is mathematically distasteful. See tables 6 and 7.

Supposing the arbitrary choice of $N = 0$ to be at fault, we have next tried $N = 1$ for the case $\delta = 1$ with 10.2 as lowest approximation and rigorous application of formula 9.7 or 10.5. Convergence resulted everywhere and for the particular example considered <u>the method thus appears thoroughly satisfactory if one takes $N = 1$</u>.

(It may be remarked that the values for $g_{\delta=1}$ obtained by the two different methods differ by about one per cent - this discrepancy is ascribed solely to the inaccuracy of trapezoidal integration since the integrands involved differ considerably.

11. ALTERNATIVE METHOD WHEN $\delta \neq 1$.

In this section only we adopt the following notation:

$$\varepsilon = 1 - \delta \quad , \qquad 11.1$$

$$q_0 = q_{\delta = 1} \quad , \qquad 11.2$$

$$g_0 = g_{\delta = 1} \quad , \qquad 11.3$$

$$\zeta_0 = \int^{\tau} \frac{1}{g_0}\, d\tau \quad . \qquad 11.4$$

In order to solve equations 7.19, 7.21,

$$g\left(1 - \frac{dq}{d\tau}\right) = q\left(\mathcal{R}_0 + \mathcal{R}_1 \xi + \mathcal{R}_2 \xi^2\right) , \qquad 11.5$$

$$(1-\varepsilon)\, g \frac{d\xi}{d\tau} - \xi = -\varepsilon\, k\, g \quad , \qquad 11.6$$

we assume the expansions

$$q = \frac{q_0}{1 + \varepsilon q_1 + \varepsilon^2 q_2 + \cdots} \quad , \qquad 11.7$$

$$g = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \cdots \quad , \qquad 11.8$$

$$\xi = \varepsilon \xi_1 + \varepsilon^2 \xi_2 + \cdots \quad , \qquad 11.9$$

where the suffixed quantities are independent of ε (i.e. of δ), substitute them into 11.5, 11.6, and equate coefficients of powers of ε. The solution for the case $\varepsilon = 0$ is supposed known; we abbreviate

$$\gamma_1 = \frac{g_1}{g_0} \, , \quad \gamma_2 = \frac{g_2}{g_0} , \cdots ; \qquad 11.10$$

the resulting <u>linear</u> equations are now easily integrated in the following order:

$$\xi_1 = k\, e^{\zeta_0} \int_{\tau}^{\tau_0} e^{-\zeta_0}\, d\tau \quad , \qquad 11.11$$

$$g_1 = q_0 \frac{1}{g_0}\, e^{\zeta_0} \left[-q_1 \int_{\tau}^{\tau_0} e^{-\zeta_0} \mathcal{R}_0\, d\tau + \int_{\tau}^{\tau_0} e^{-\zeta_0} \mathcal{R}_1 \xi_1\, d\tau \right] , \qquad 11.12$$

where

$$q_1 = \int_{\tau_c}^{\tau_0} e^{-\zeta_0} \mathcal{R}_1 \xi_1\, d\tau \Big/ \int_{\tau_c}^{\tau_0} e^{-\zeta_0} \mathcal{R}_0\, d\tau \quad . \qquad 11.13$$

$$\xi_2 = \xi_1 - e^{\zeta_0}\int_\tau^{\tau_\infty} e^{-\zeta_0}(1-\gamma_1)\frac{\xi_1}{g_0}\,d\tau \qquad 11.14$$

$$g_2 = q_0\frac{1}{g_0}e^{\zeta_0}\Bigg(-q_2\int_\tau^{\tau_\infty} e^{-\zeta_0}\mathcal{R}_0\,d\tau$$

$$+\int_\tau^{\tau_\infty} e^{-\zeta_0}\Big[(q_1^2+q_1\gamma_1+\gamma_1^2)\mathcal{R}_0+\mathcal{R}_1\{\xi_2-(q_1+\gamma_1)\xi_1\}+\mathcal{R}_2\xi_1^2\Big]d\tau\Bigg) \qquad 11.15$$

where

$$q_2 = \frac{\int_{\tau_0}^{\tau_\infty} e^{-\zeta_0}\Big[(q_1^2+q_1\gamma_1+\gamma_1^2)\mathcal{R}_0+\mathcal{R}_1\{\xi_2-(q_1+\gamma_1)\xi_1\}+\mathcal{R}_2\xi_1^2\Big]d\tau}{\int_{\tau_0}^{\tau_\infty} e^{-\zeta_0}\mathcal{R}_0\,d\tau}\,; \qquad 11.16$$

$$\xi_3 = \xi_2 - e^{\zeta_0}\int_\tau^{\tau_\infty} e^{-\zeta_0}\Big\{(1-\gamma_1)\frac{\xi_2}{g_0}+(\gamma_1^2-\gamma_2)\frac{\xi_1}{g_0}\Big\}d\tau \qquad 11.17$$

$$g_3 = q_0\frac{1}{g_0}e^{\zeta_0}\Bigg(-q_3\int_\tau^{\tau_\infty} e^{-\zeta_0}\mathcal{R}_0\,d\tau$$

$$+\int_\tau^{\tau_\infty} e^{-\zeta_0}\Big[\{q_1(2q_2-q_1^2)+\gamma_2(q_1+\gamma_1)+\gamma_1(q_2+\gamma_2)-\gamma_1(q_1^2+q_1\gamma_1+\gamma_1^2)\}\mathcal{R}_0$$

$$+\mathcal{R}_1[\{q_1^2+\gamma_1(q_1+\gamma_1)-(q_2+\gamma_2)\}\xi_1-(q_1+\gamma_1)\xi_2+\xi_3]$$

$$+\mathcal{R}_2\{2\xi_1\xi_2-(q_1+\gamma_1)\xi_1^2\}\Big]d\tau\Bigg) \qquad 11.18$$

where

$$q_3 = \frac{\int_{\tau_0}^{\tau_\infty}(\text{same integrand as in second integral of 11.18})\,d\tau}{\int_{\tau_0}^{\tau_\infty} e^{-\zeta_0}\mathcal{R}_0\,d\tau}\,; \qquad 11.19$$

etc. See graph 22, p. 53.

12. NEGLECT OF BACK REACTION; VARIATION OF HOT BOUNDARY TEMPERATURE

Neglect of the back reaction. It was desirable to show that the method is unaffected, and the results little changed, if the back-reaction is neglected. In this case, if one considers only $\delta = 1$, cf. A.4,

$$\mathcal{R}_o(\tau) = (x^*)^2 \varphi(\tau) \qquad 12.1$$

$$= \left(\frac{\tau_\infty - \tau}{\tau_\infty - \tau_o}\right)^2 \varphi(\tau) \qquad . \qquad 12.2$$

Here τ_∞ and τ_∞^* coincide, and $x_\infty = 0$. As lowest approximation one takes

$$g^{(0)} = \frac{1}{(\tau_\infty - \tau_o)^2} (\tau - \tau_o)(\tau_\infty - \tau)^2 . \qquad 12.3$$

One finds the expected results, but very near τ_∞, when 10.5 is used there is again no convergence and formula 10.6 has to be applied; use of $N = \frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, give consistent results. Cf. graph 24.

Variation of the hot boundary temperature. It is also of interest to investigate how the eigenvalue parameter q changes with τ_∞ when τ_o is kept constant. The procedure outlined just above was thus repeated for different τ_∞ with formulae 12.2 and 12.3. The results are shown in graphs 23 and 24.

APPENDIX

Simple chain reaction flame. Let us consider an idealized flame based on the reactions

$$A \rightleftarrows B \;, \tag{A.1}$$

$$A + B \rightleftarrows B + C \;, \tag{A.2}$$

and let us suppose that to a first approximation component B is in equilibrium with A, so that

$$x_B = \varphi(\tau)\, x_A \;. \tag{A.3}$$

The rate of decrease of component A (which we have roughly referred to as the reaction rate) may be assumed as proportional to

$$x_B x_A - x_B(1 - x_A - x_B)\,\Phi(\tau) \;, \tag{A.4}$$

that is,

$$x_A\,\varphi(\tau)\left[x_A - \{1 - x_A - x_A\,\varphi(\tau)\}\,\Phi(\tau)\right] . \tag{A.5}$$

Dropping the suffix A, we see that the problem is equivalent to that of a unimolecular reaction of the type already considered*, with the total reaction rate of the fuel component of the second order and proportional to

$$R(x,\tau) = x\Big(x\left[\varphi(\tau) + \varphi(\tau)\,\Phi(\tau) + \{\varphi(\tau)\}^2\,\Phi(\tau)\right] - \varphi(\tau)\,\Phi(\tau)\Big) . \tag{A.6}$$

If at the hot boundary temperature τ_∞ a fraction x_∞ of the fuel is left,

$$x_\infty = \frac{\Phi(\tau_\infty)}{1 + \Phi(\tau_\infty) + \varphi(\tau_\infty)\,\Phi(\tau_\infty)} \tag{A.7}$$

We note that x is a factor of the total reaction rate $R(x,\tau)$,

$$R(0,\tau) = 0 \tag{A.8}$$

Thus, in terms of a unimolecular reaction one could say that the fuel acts at the same time as a catalyst whose presence is indispensable.

---

* This is not quite true: there is an additional small term in the energy eqn. 1.3. The full ABC problem will be considered in a forthcoming report.

Particular example. We take

$$\Phi(\tau) = \varphi(\tau) . \qquad A.9$$

In the original formulation of the problem $\varphi(\tau)$ was taken to be equal to $e^{-1/\tau}$; in the present treatment we have put

$$\varphi(\tau) = \kappa e^{-1/\tau} , \qquad A.10$$

and chosen the constant κ so as to get simple expressions for λ_∞ and the linear function χ^* of 7.7 (or 7.10). Thus with

$$\tau_0 = .02 , \qquad A.11$$

$$\tau_\infty = .218 , \qquad A.12$$

$$\tau_\infty^* = .220 , \qquad A.13$$

so that, cf. 7.8, $b = 5$, and by A.7,

$$\chi_\infty = \frac{1}{100} = \frac{\kappa e^{-1/\tau_\infty}}{1 + \kappa e^{-1/\tau_\infty} + (\kappa e^{-1/\tau_\infty})^2} , \qquad A.14$$

which gives

$$\kappa = .992\ 168\ 9 \qquad A.15$$

On comparison of A.6 and 7.15 one finds

$$\mathcal{R}_2(\tau) = \varphi(\tau) + \{\varphi(\tau)\}^2 + \{\varphi(\tau)\}^3 \qquad A.16$$

$$\mathcal{R}_1(\tau) = 2\chi^* \mathcal{R}_2(\tau) - \{\varphi(\tau)\}^2 \qquad A.17$$

and $\mathcal{R}_0(\tau)$ is given by 8.3.

Acknowledgments.

Without many constructive discussions with Professor J. O. Hirschfelder, Professor C. F. Curtiss, and Dr. E. S. Campbell, the author would have found himself in a succession of blind alleys. If the present method should eventually prove to be a satisfactory one for a mathematical description of flames, it will in no small measure have been based on their experience, so readily shared.

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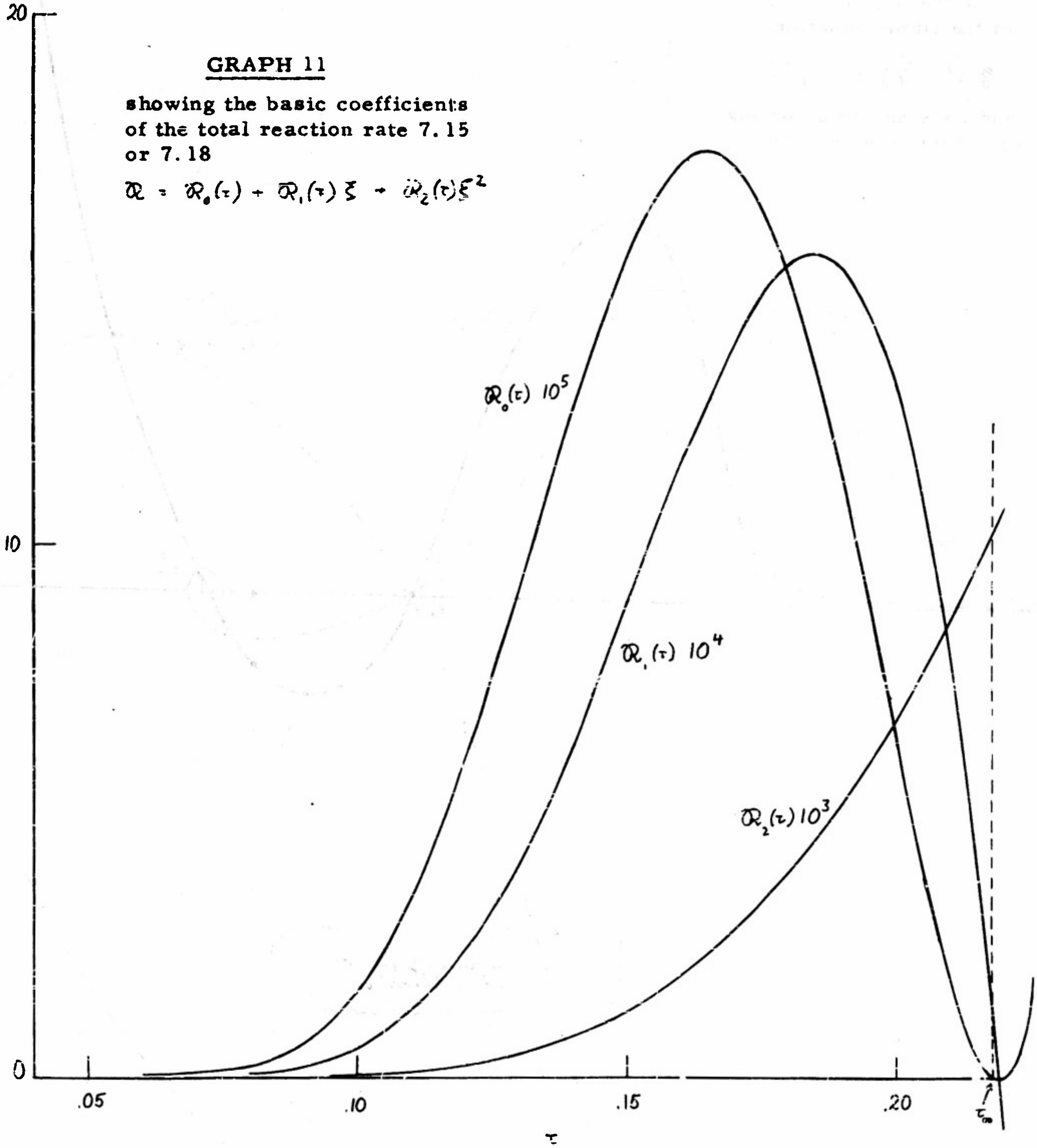
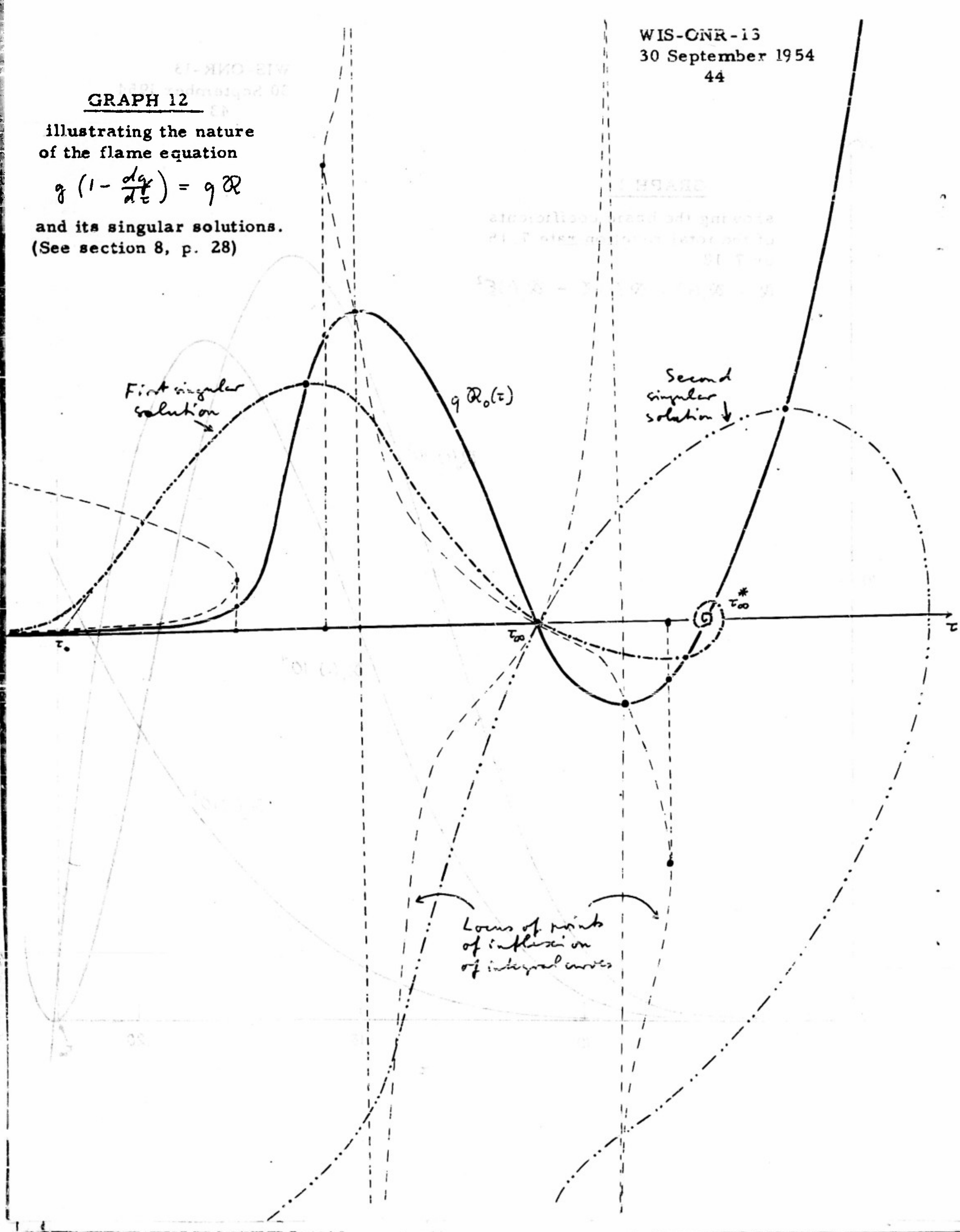

20
10
0
.05
.10
.15
.20
τ
τ_∞
GRAPH 11
showing the basic coefficients of the total reaction rate 7.15 or 7.18
$\mathcal{R} = \mathcal{R}_0(\tau) + \mathcal{R}_1(\tau)\xi + \mathcal{R}_2(\tau)\xi^2$
$\mathcal{R}_0(\tau)\ 10^5$
$\mathcal{R}_1(\tau)\ 10^4$
$\mathcal{R}_2(\tau)\ 10^3$

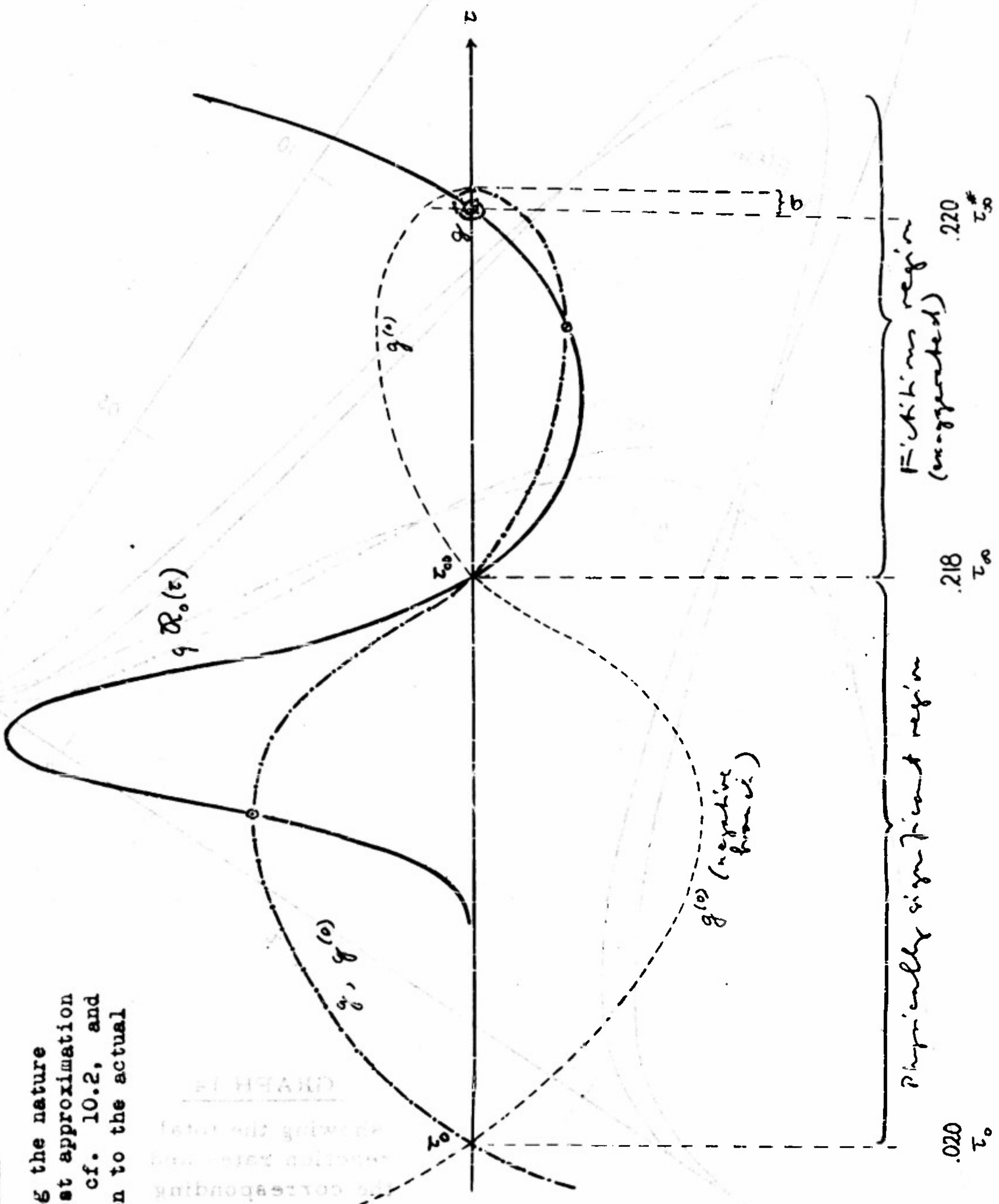
GRAPH 12

illustrating the nature of the flame equation

$$q\left(1-\frac{dq}{d\tau}\right)=q\mathcal{R}$$

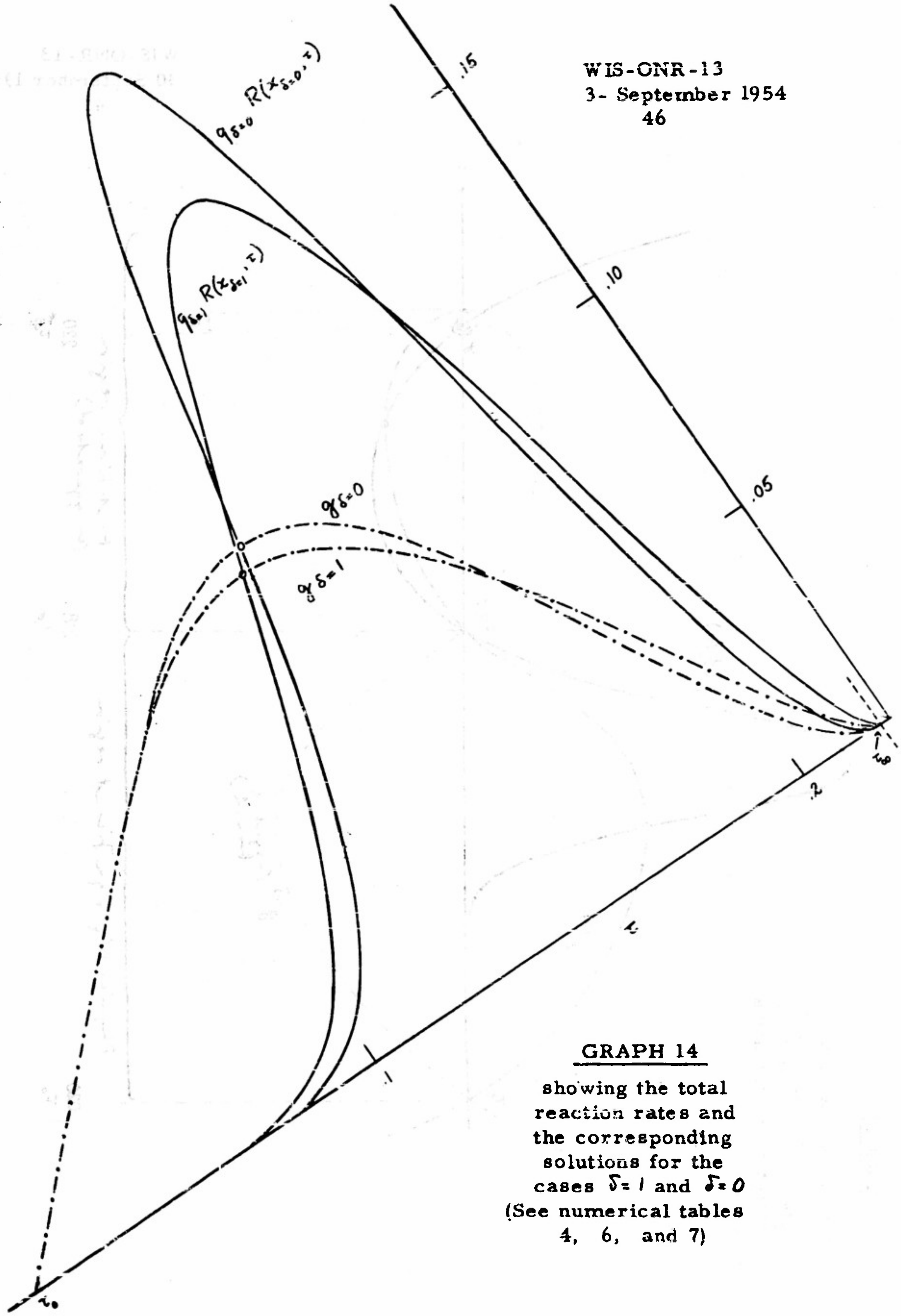
and its singular solutions. (See section 8, p. 28)





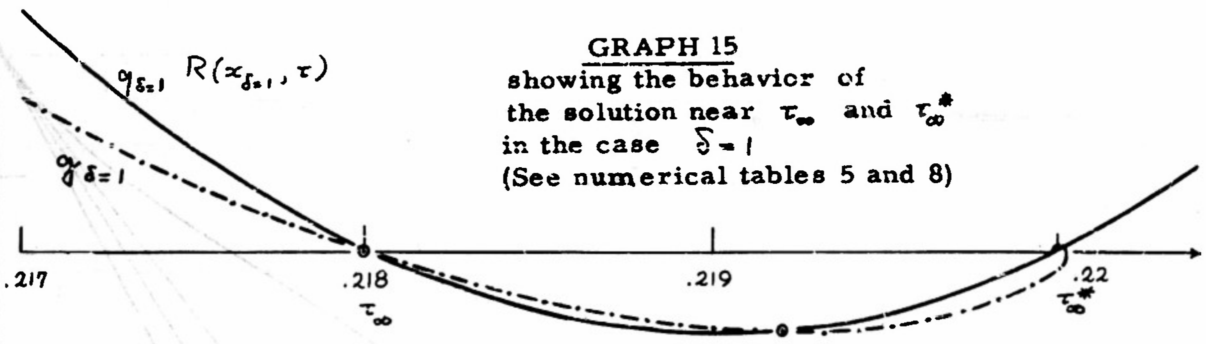
GRAPH 13.

illustrating the nature of the lowest approximation $g^{(0)}$, cf. 10.2, and its relation to the actual solution.

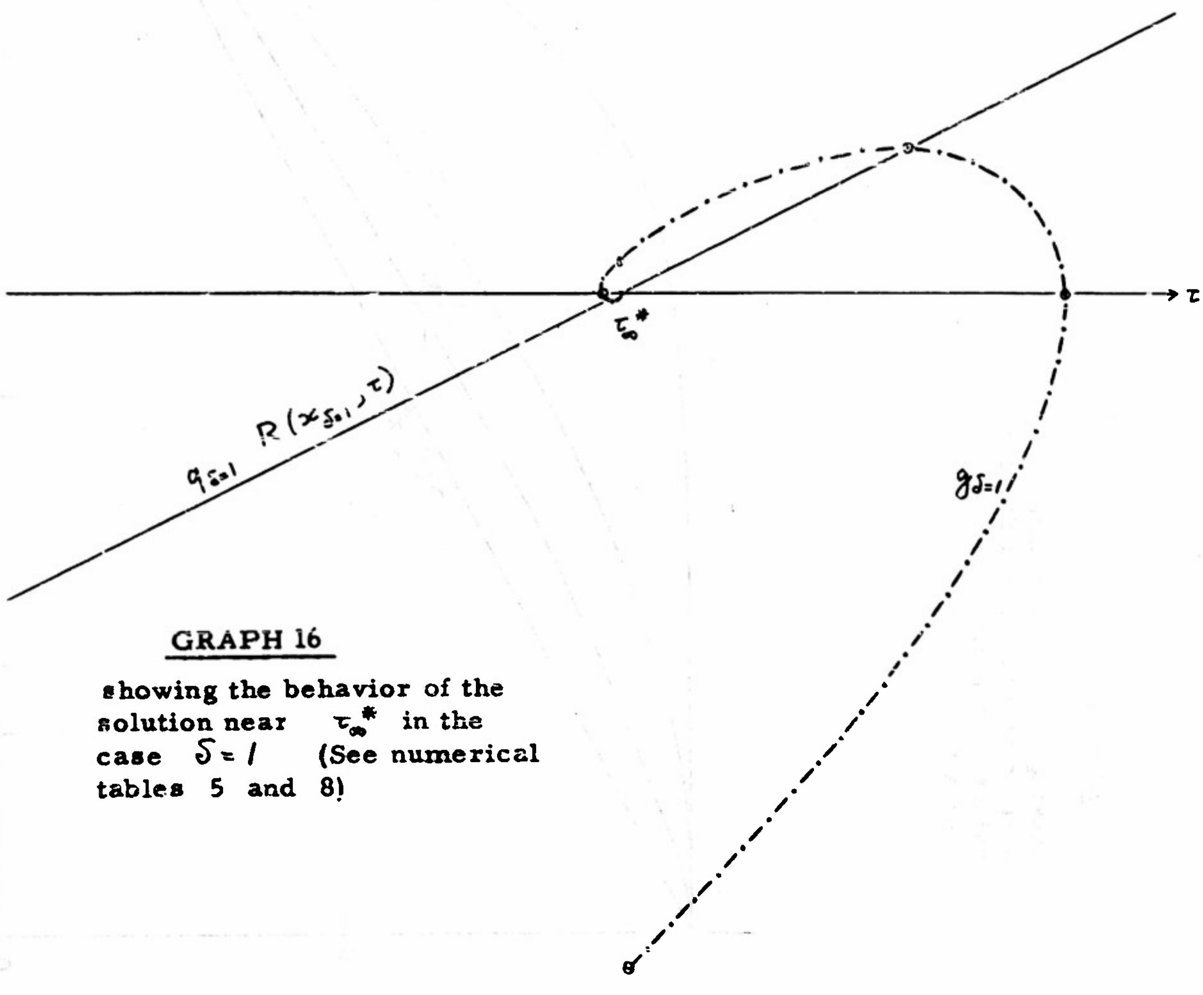


GRAPH 14

showing the total reaction rates and the corresponding solutions for the cases $\delta = 1$ and $\delta = 0$ (See numerical tables 4, 6, and 7)



GRAPH 15
showing the behavior of the solution near τ_∞ and τ_∞^* in the case $\delta = 1$
(See numerical tables 5 and 8)

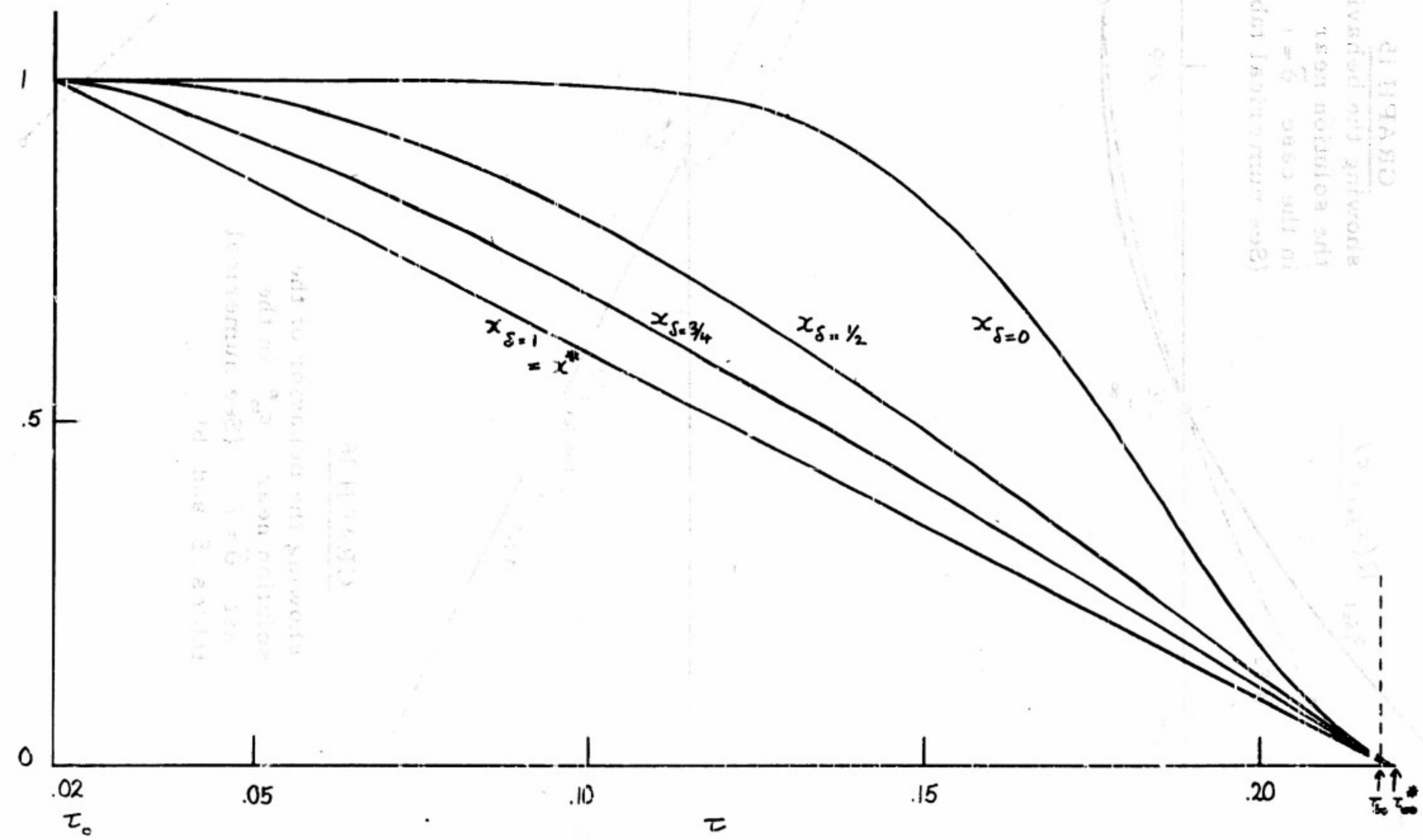


GRAPH 16

showing the behavior of the solution near τ_∞^* in the case $\delta = 1$ (See numerical tables 5 and 8)

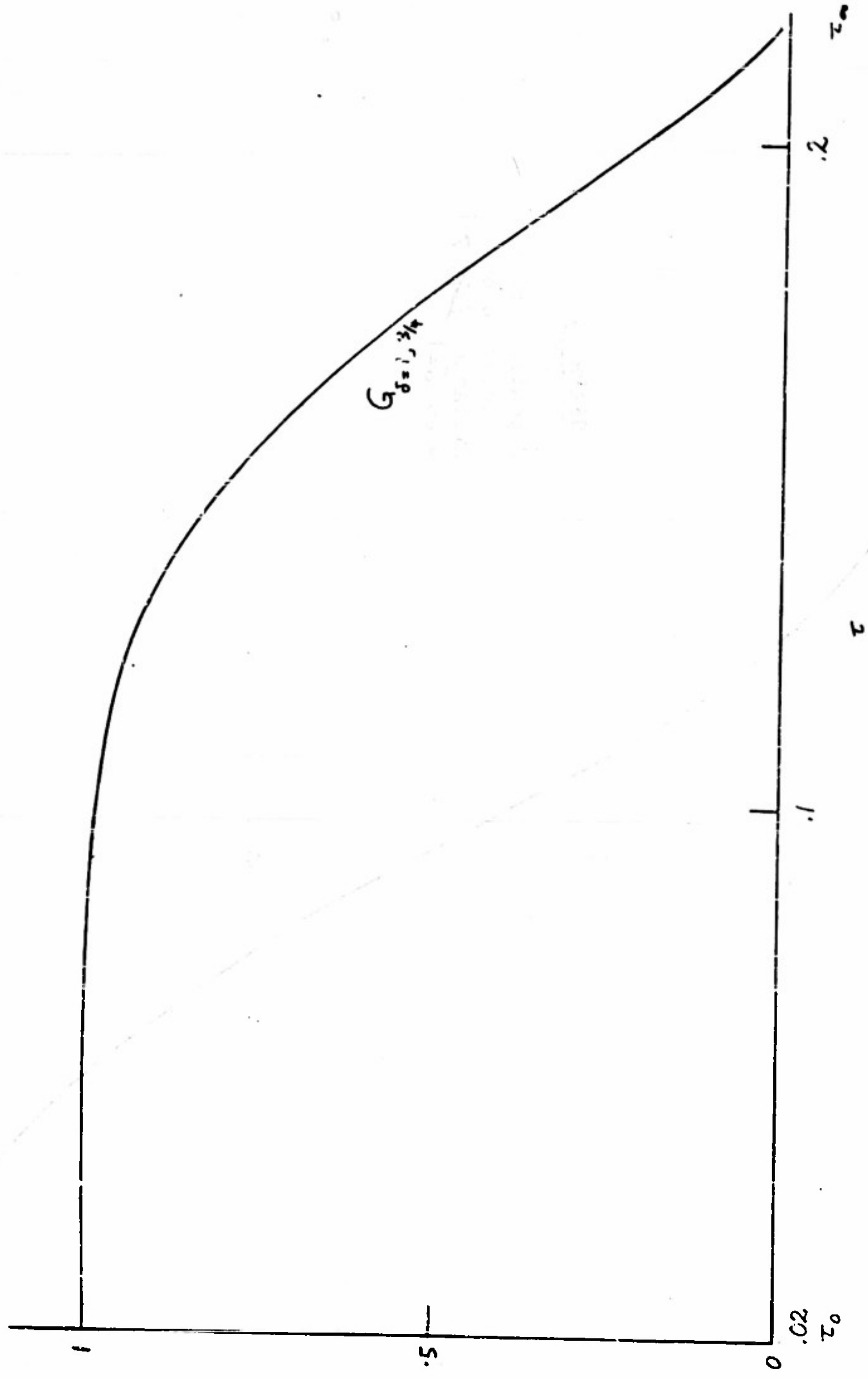
GRAPH 17

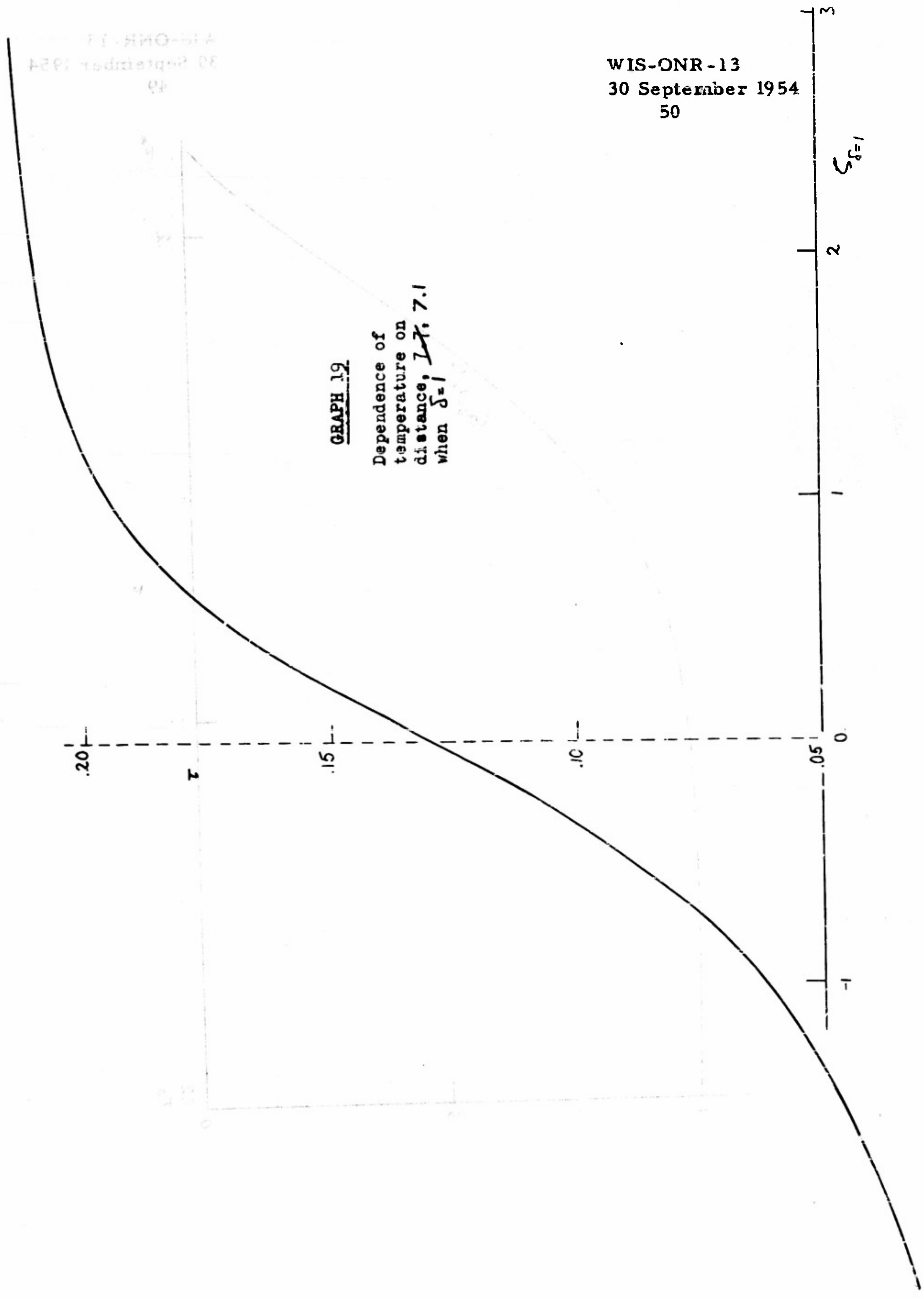
showing the fuel decrease with temperature for different values of the diffusion constant.



GRAPH 18

showing the behavior of the fractional mass rate of flow, cf. 1.12 and numerical table 6.



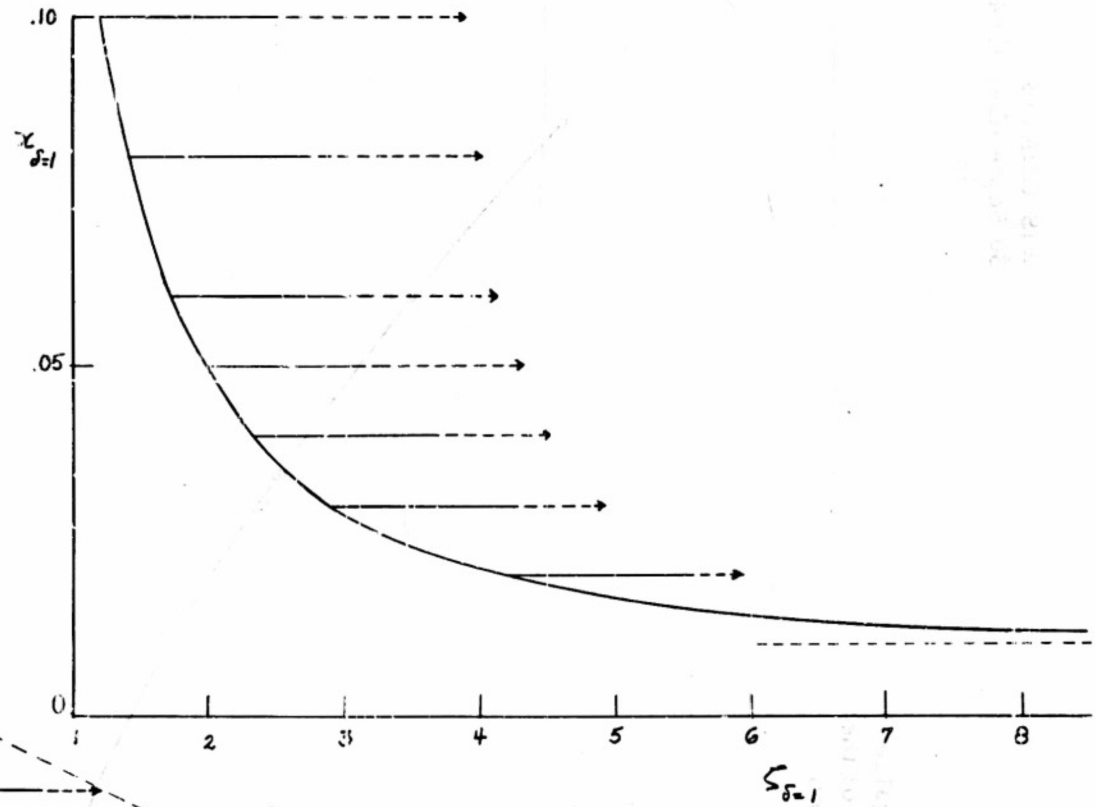


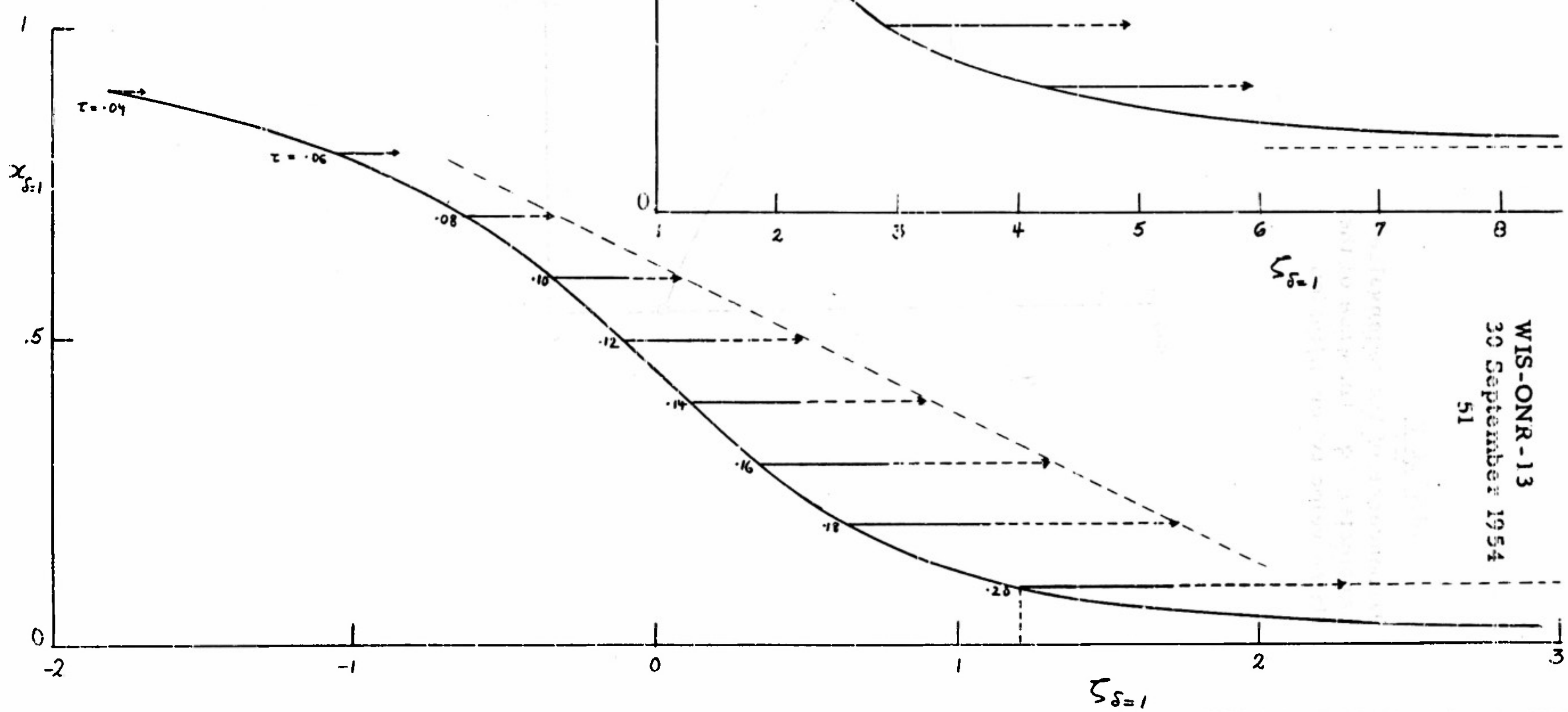
GRAPH 19

Dependence of temperature on distance, I-7, 7.1 when $\delta=1$

GRAPH 20

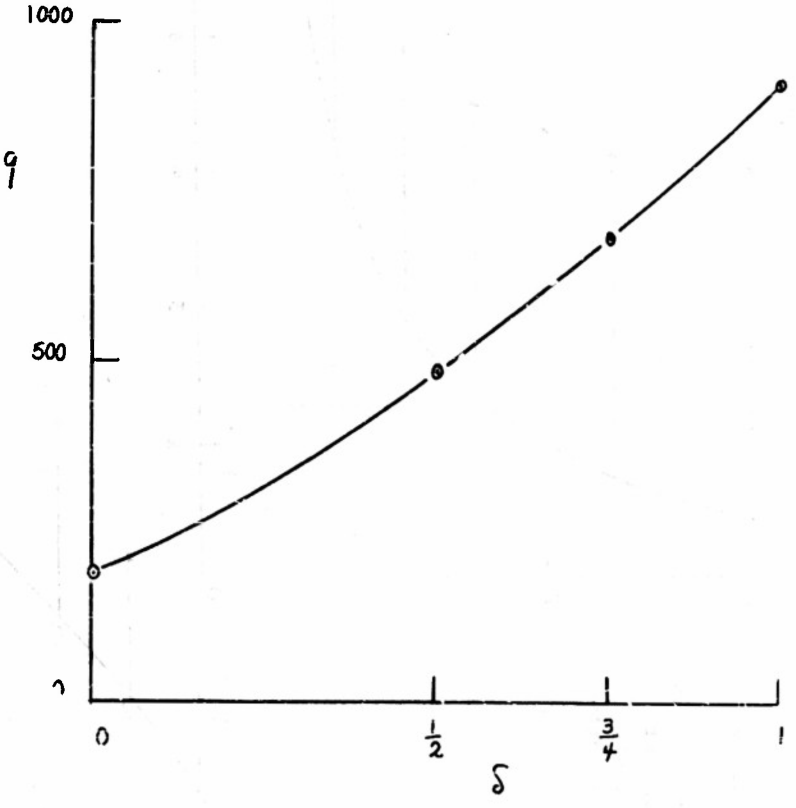
Variation of fuel component
with distance when $\delta = 1$
The arrows represent velocities
of fuel on the scale
(vel. at cold boundary) = 1/10 in.
The end portions of the arrows
represent the velocities over
and above the mean mass velocity
of the mixture.





GRAPH 21

Dependence of the eigenvalue parameter q (inverse of the flame velocity) on diffusion



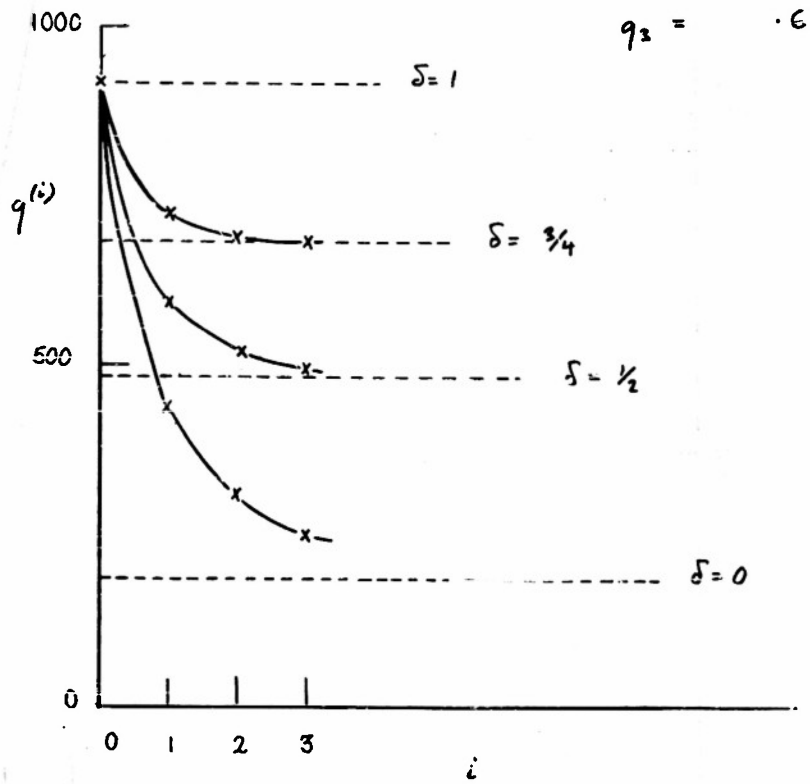
GRAPH 22

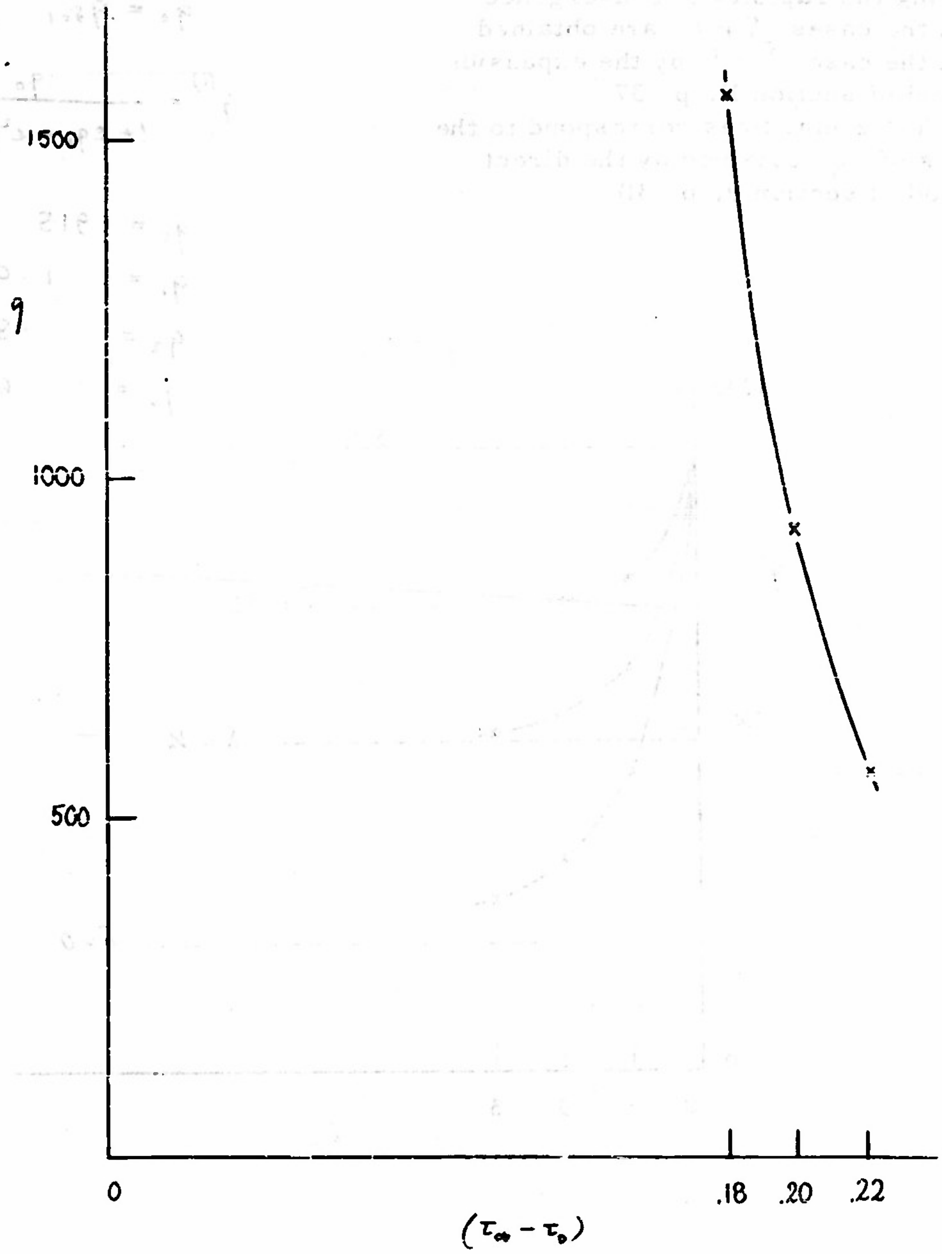
showing the rapidity of convergence when the cases $\delta \neq 1$ are obtained from the case $\delta = 1$ by the expansion method of section 11, p. 37
(The horizontal lines correspond to the values of q obtained by the direct method of section 9, p. 31)

$$q_0 = q_{\delta=1} \quad , \quad \varepsilon = 1-\delta$$

$$q^{(i)} = \frac{q_0}{1+\varepsilon q_1 + \varepsilon^2 q_2 + \ldots + \varepsilon^i q_i}$$

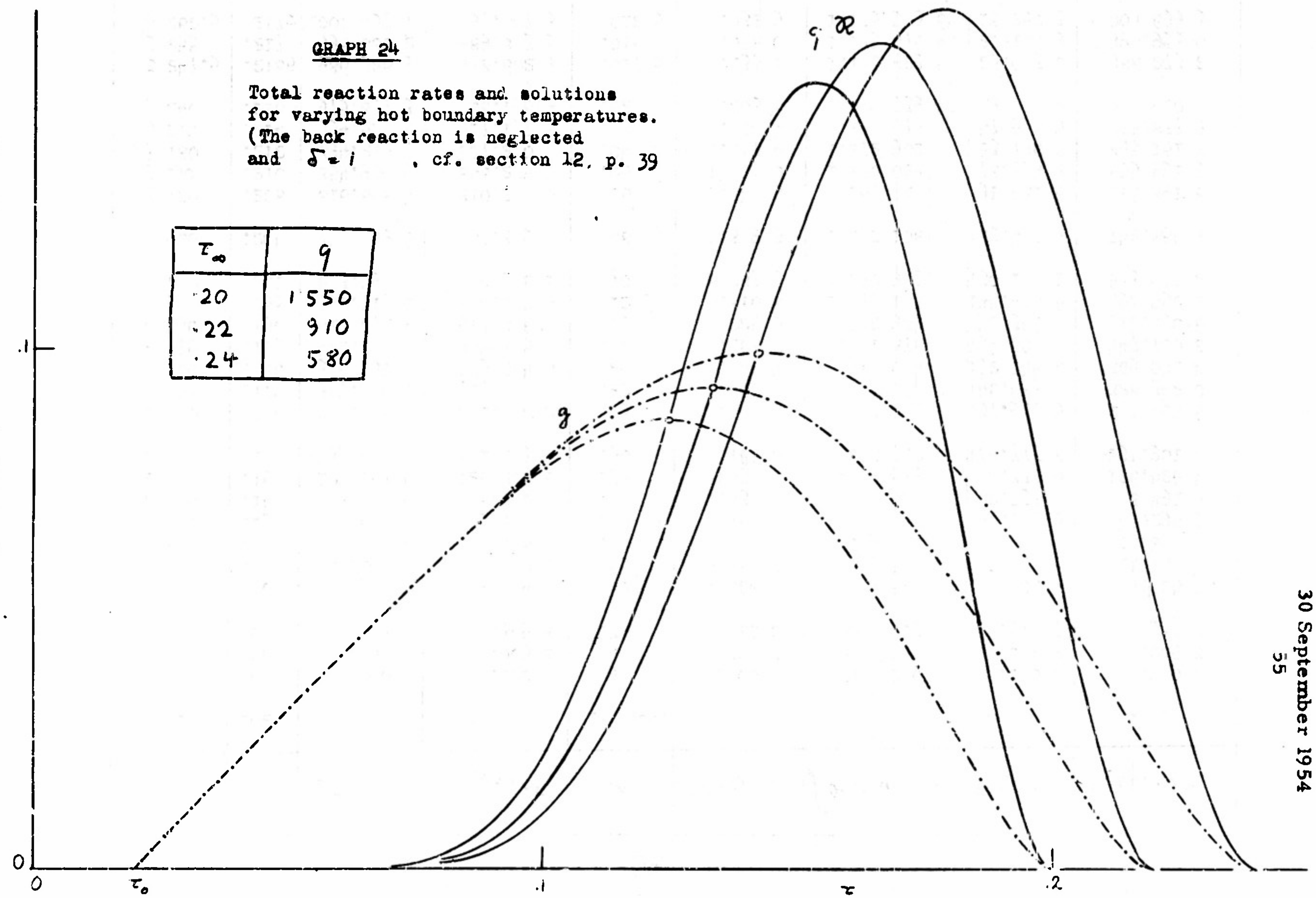
$q_0 =$ 915
$q_1 =$ 1.08
$q_2 =$ .88
$q_3 =$ .65





GRAPH 23

Dependence of the parameter q on the hot boundary temperature. (The back reaction is neglected, $\delta = 1$, and the cold boundary temperature is kept constant; Cf. section 12, p. 39)



GRAPH 24

Total reaction rates and solutions for varying hot boundary temperatures. (The back reaction is neglected and $\delta = 1$, cf. section 12, p. 39

τ_∞	q
.20	1550
.22	910
.24	580

TABLE 4

T	τ	$e^{-\frac{1}{\tau}} \times 10^5$	$\varphi(\tau) \times 10^5$	x^*	$\mathcal{R}_0(\tau) \times 10^5$	$\int_{\tau_0}^{\tau} \mathcal{R}_0(\tau)\, d\tau \times 10^5$	$\mathcal{R}_1(\tau) \times 10^5$	$\mathcal{R}_2(\tau) \times 10^5$
300	.02		-	1 (DEF.)	-	0	-	-
600	.04	.000 0	.000 0	.9	.000 0	.000 000	.000 0	.000 0
900	.06	.005 8	.005 8	.8	.003 7	.000 025	.009 2	.005 8
1 200	.08	.372 7	.369 8	.7	.181 2	.001 003	.517 7	.369 9
1 500	.10	4.540 0	4.504 5	.60	1.621 6	.014 827	5.405 4	4.504 7
1 650	.11	11.26 5	11.180 4	.55	3.381 7	.039 178	12.298 5	11.181 7
1 800	.12	24.036 7	23.848 5	.50	5.960 7	.085 191	23.848 5	23.854 2
1 950	.13	45.632 4	45.275 0	.45	9.163 1	.160 470	40.745 5	45.295 5
2 100	.14	79.049 0	78.430 0	.40	12.534 0	.269 016	62.731 7	78.491 5
2 250	.15	127.263 4	126.266 8	.35	15.431 4	.409 573	88.339 0	126.426 4
2 400	.16	193.045 4	191.533 6	.30	17.161 0	.573 752	114.773 8	191.901 2
2 550	.170	278.821 7	276.638 2	.25	17.146 5	.747 043	137.937 5	277.405 6
2 625	.175	329.850 6	327.267 5	.22 5	16.381 3	.831 088	146.683 0	328.342 0
2 700	.180	386.592 0	383.564 6	.20	15.107 4	.910 009	152.545 4	385.041 4
2 775	.185	449.223 3	445.705 4	.17 5	13.363 2	.981 378	154.708 7	447.700 8
2 850	.190	517.892 4	513.836 7	.15	11.225 0	1.042 984	152.306 9	516.490 6
2 925	.195	592.719 0	588.077 4	.12 5	8.810 8	1.093 168	144.430 6	591.556 1
3 000	.200	673.794 7	668.518 1	.10	6.283 3	1.130 903	130.134 2	673.017 2
3 060	.204	743.199 6	737.379 5	.08	4.319 3	1.152 108	113.419 9	742.856 9
3 120	.208	816.668 4	810.273 0	.06	2.546 9	1.165 713	91.461 6	816.891 6
3 150	.210	854.930 9	848.235 9	.05	1.779 0	1.170 024	78.354 2	855.491 9
3 180	.212	894.213 3	887.210 6	.04	1.117 4	1.172 900	63.740 8	895.151 8
3 210	.214	934.515 6	927.197 3	.03	.584 4	1.174 579	47.555 5	935.873 9
3 240	.216	975.837 3	968.195 4	.02	.203 6	1.175 339 42	29.732 4	977.660 2
3 247.5	.2165	986.326 9	978.602 9	.017 5	.135 1	1.175 423 64	25.013 0	988.273 2
3 255	.217	996.880 2	989.073 5	.015	.078 0	1.175 476 40	20.185 9	998.953 0
3 262.5	.2175	1007.497 1	999.607 3	.012 5	.032 9	1.175 503 64	15.250 3	1 009.699 3
3 270	.218	1018.177 5	1 010.204 1	.01 (DEF.)	0	1.175 511 35	10.205 1	1 020.512 3
3 300	.220	1061.534 7	1 053.221 6	0 (DEF.)	0			

$\varphi(\tau) = \kappa e^{-1/\tau}$, $\kappa = .992\ 168\ 9$

TABLE 5

T	τ	$e^{-\frac{1}{\tau}} \times 10^5$	$\varphi(\tau) \times 10^5$	x^*	$\mathcal{R}_0(\tau) \times 10^5$	$\int_{\tau}^{\tau_\infty} \mathcal{R}_0(\tau)\, d\tau \times 10^5$	$\mathcal{R}_1(\tau) \times 10^5$	$\mathcal{R}_2(\tau) \times 10^5$
3 255	.217	996.880 1	989.073 4	.015	.078 02	.000 035 08	20.185 5	998.952 9
	.217 2	1 001.119 2	993.279 3	.014	.058 51	.000 021 42	18.224 8	1 003.243 3
	.217 4	1 005.368 5	997.495 4	.013	.040 93	.000 011 48	16.246 2	1 007.544 7
	.217 6	1 009.628 0	1 001.721 5	.012	.025 29	.000 004 86	14.250 1	1 011.856 5
	.217 8	1 013.897 6	1 005.957 7	.011	.011 64	.000 001 16	12.236 5	1 016.179 0
3 270	.218	1 018.177 5	1 010.204 1	.01	0 *	0	10.205 1	1 020.512 3
	.218 2	1 022.467 4	1 014.460 4	.009	-.009 61	.000 000 96	8.156 1	1 024.856 1
	.218 4	1 026.767 6	1 018.726 9	.008	-.017 15	.000 003 64	6.089 4	1 029.210 6
	.218 6	1 031.077 9	1 023.003 4	.007	-.022 61	.000 007 61	4.004 6	1 033.575 9
	.218 8	1 035.398 4	1 027.290 1	.006	-.025 95	.000 012 47	1.902 2	1 037.951 7
3 285	.219	1 039.729 0	1 031.586 8	.005	-.027 35	.000 017 78	- .218 3	1 042.338 3
	.219 2	1 044.069 8	1 035.893 6	.004	-.026 18	.000 023 11	- 2.357 0	1 046.735 6
	.219 4	1 048.420 8	1 040.210 5	.003	-.023 00	.000 028 03	- 4.513 6	1 051.143 5
	.219 6	1 052.781 9	1 044.537 5	.002	-.017 60	.000 032 09	- 6.688 4	1 055.562 1
	.219 8	1 057.153 2	1 048.874 5	.001	-.009 94	.000 034 84	- 8.881 4	1 059.991 3
3 300	.220	1 061.534 6	1 053.221 6	0	0 **	.000 035 84	-11.092 8	1 064.431 2
	.220 2	1 065.926 2	1 057.578 8	-.001	.012 25		-13.322 5	1 068.881 8
	.220 4	1 070.328 0	1 061.946 2	-.002	.026 85		-15.[illegible] 7	1 073.343 3
	.220 6	1 074.739 8	1 066.323 4	-.003	.043 81		-17.337 3	1 077.815 1

$$* \left[\frac{d\mathcal{R}_0(\tau)}{d\tau}\right]_{\tau_\infty} = -.000\ 531\ 5 \qquad \left[\frac{d^2\mathcal{R}_0(\tau)}{d\tau^2}\right]_{\tau_\infty} = .508\ 6$$

$$** \left[\frac{d\mathcal{R}_0(\tau)}{d\tau}\right]_{\tau_\infty^*} = .000\ 554\ 6$$

TABLE 6

	$\delta = 1$; (q = 914.6[illegible])				$\delta = 3/4$; (q = 683.05)			
τ	g	ζ	x (= x^*)	G	g	ζ	x	G
.02	0	$-\infty$	1 (DEF.)	1	0	$-\infty$	1	1
.04	.020 00	-1.808 6	.9	1.0[illegible] 0	.020 00	-1.809 0	.950 38	1.000 00
.06	.039 99	-1.058 6	.8	[illegible] 9	[illegible]	-1.058 9	.874 99	.999 95
.08	.059 81	-.641 3	.7	.999 1	.059 79	-.641 5	.785 06	.998 95
.10	.077 96	-.345 8	.60	.989 8	.07[illegible] 89	-.346 0	.684 79	.989 45
.11	.085 24	-.223 0	.55	.976 2	.085 15	-.223 1	.631 72	.975 95
.12	.090 44	-.103 1	.50	.952 2	.090 42	.109 1	.577 11	.952 10
.13	.092 94 (M)	0 (DEF.)	.45	.914 7	.092 90	[illegible]	.521 22	.914 50
.14	.092 20	.108 0	[illegible]	.861 0	.092 13	.108 1	.464 29	.860 65
.15	.087 94	.219 1	.35	.789 7	.087 75	.219 3	.406 56	.788 75
.16	.080 [illegible]	.338 4	.30	.700 3	.079 71	.339 0	.348 24	.698 55
.170	.068 76	.473 6	.25	.593 8	.068 23	.475 0	.289 52	.591 15
.175	.062 01	.550 3	.225	.535 0	.061 36	.552 4	.260 02	.531 80
.180	.054 63	.636 3	.20	.473 2	.053 89	.639 6	.230 50	.469 45
.185	.046 75	.735 6	.175	.408 8	.045 92	.740 4	.200 95	.404 60
.190	.038 54	.853 9	.15	.342 7	.037 66	.861 2	.171 42	.338 30
.195	.030 18	1.001 6	.125	.275 9	.029 30	1.012 9	.141 96	.271 50
.200	.021 94	1.198 4	.10	.209 7	.021 11	1.216 7	.112 62	.205 55
.204	.015 [illegible]	1.418	.08	.158 1	.014 89	1.44[illegible]	.089 28	.154 45
.208	.009 83	1.75	.06	.109 1	.009 24	1.80	.066 19	.106 20
.210	.007 19	1.99	.05	.086 0	.00[illegible]	2.05	.056 63	.083 50
.212	.004 80	2.34	.04	.064 0	.00[illegible]	2.43	.043 26	.062 10
.214	.002 73	2.91	.03	.043 6	.002 [illegible]9	3.06	.032 03	.042 45
.216	.001 09	4.20	.02	.025 5	.000 95	4.51	.020 87	.024 75
.216 5	.000 75	4.8	.017 5	.021 [illegible]	.000 66	5.[illegible]	.018 18	.020 80
.217	.000 46	5.6	.015	.017 3	.000 41	6.1	.015 47	.017 05
.217 5	.000 20	7.4	.012 5	.013 5	.000 19	8.1	.012 92	.013 45
.218	0	∞	.01 (DEF.)	.01	0	∞	.01	.01

TABLE 7

	$\delta = \frac{1}{2}$; $(q = 484)$				$\delta = 0$; $(q = 187)$		
τ	g	ζ	x	G	g	ζ	x, G
.02	0	$-\infty$	1	1	0	$-\infty$	1
.04	.020 00	-1.808 6	.995 0	1.000 0	.020 00	-1.798 2	1.000 0
.06	.039 99	-1.058 5	.951 7	1.000 0	.039 97	-1.048 0	.999 9
.08	.059 79	- .641 2	.884 4	.999 0	.059 87	- .630 8	.999 3
.10	.077 90	- .345 8	.792 6	.989 5	.078 69	- .336 7	.993 5
.11	.085 27	- .222 8	.739 4	.976 4	.086 97	- .215 7	.984 8
.12	.090 57	- .108 9	.681 6	.952 8	.093 56	- .104 7	.967 8
.13	.093 1	0	.619 6	.916	.097 47	0	.937 4
.14	.092 3	.107 9	.554 4	.861	.097 7	.102 5	.888
.15	.087 8	.219 0	.486 6	.789	.093 4	.207 2	.817
.16	.079 5	.338 8	.416 8	.697	.084 0	.320 3	.720
.170	.067 7	.475 6	.345 6	.589	.069 8	.451 4	.599
.175	.060 7	.553 7	.309 7	.528	.061 1	.528 2	.530
.180	.053 1	.641 9	.273 6	.466	.051 7	.617 5	.459
.185	.045 0	.744 5	.237 6	.400	.042 1	.725 2	.385
.190	.036 6	.868 4	.201 6	.333	.032 5	.861 5	.313
.195	.028 3	1.025	.166 0	.266	.023 5	1.045	.242
.200	.020 1	1.238	.131 2	.201	.015 38	1.314	.176 9
.204	.014 0	1.408	.099 8	.150	.009 89	1.646	.129 5
.208	.008 6	1.86	.074 0	.103	.005 43	2.22	.087 1
.210	.006 2	2.13	.060 3	.081	.003 68	2.67	.068 4
.212	.004 08	2.54	.047 6	.060 4	.002 24	3.39	.051 2
.214	.002 26	3.23	.035 4	.041 3	.001 15	4.71	.035 8
.216	.000 83	4.88	.025 2	.024 2	.000 38	8.21	.021 9
.216 5	.000 58	5.6	.019 1	.020 4	.000 25	9.9	.018 7
.217	.000 37	6.7	.016 3	.016 9	.000 15	12.5	.015 8
.217 5	.000 18	8.7	.013 8	.013 4	.000 06	18.4	.012 8
.218	0	∞	.01	.01	0	∞	.01

TABLE 8

τ	$g_{\delta=1}$	$\frac{dg_{\delta=1}}{d\tau}$	Parabolic approximation
.217	.000 460		.000 470
.217 2	.000 351		.000 358
.217 4	.000 253		.000 255
.217 6	.000 160		.000 161
.217 8	.000 076		.000 076
.218	0	−.358 0	0
.218 2	−.000 066		.000 067
.218 4	−.000 124		.000 125
.218 6	−.000 171		.000 174
.218 8	−.000 207		.000 215
.219	−.000 229		.000 246
.219 2	−.000 238 M		.000 268
.219 4	−.000 230		.000 281
.219 6	−.000 203		.000 286 M
.219 8	−.000 145		.000 281
.220	−.000 043	1	.000 267

$\theta = -\pi/2$ $\qquad$ $= r_{-\pi/2}$

	θ	$r/r_{-\pi/2}$
	− 90°	1
	− 60°	.701 0
	− 45°	.647 6
Spiral	− 30°	.628 5
near	− 15°	.632 7
ζ	0	.642 8
	15°	.602 9
	$\tan^{-1} m$	.463 0
	30°	.411 9
	45°	.195 8
	60°	.099 7
	90°	.045 2
	120°	.031 8
	150°	.028 4
	180°	.029 5
	$180° + \tan^{-1} m$	.020 9
	210°	.016 9
	240°	.0[illegible]
	270°	.002 04